Quantifiers in First Order Logic

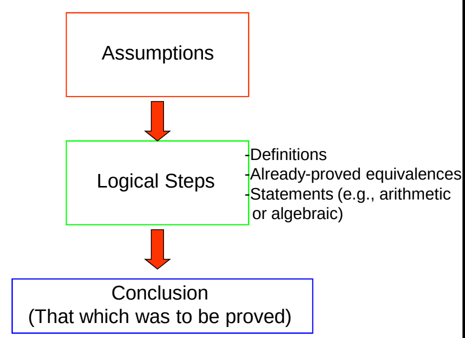
List of Logical Equivalences

p∧T ⇔ p; p∨F ⇔ p	**Identity Laws**
p∨T ⇔ T; p∧F ⇔ F	**Domination Laws**
p∨p ⇔ p; p∧p ⇔ p	**Idempotent Laws**
¬(¬p) ⇔ p	**Double Negation Law**
p∨q ⇔ q∨p; p∧q ⇔ q∧p	**Commutative Laws**
(p∨q)∨ r ⇔ p∨ (q∨r); (p∧q) ∧ r ⇔ p ∧ (q∧r)	**Associative Laws**

List of Equivalences

p∨(q∧r) ⇔ (p∨q)∧(p∨r) p∧(q∨r) ⇔ (p∧q)∨(p∧r)	**Distribution Laws**
¬(p∨q)⇔(¬p ∧ ¬q) ¬(p∧q)⇔(¬p ∨ ¬q)	**De Morgan's Laws**
	<u>Miscellaneous</u>
p ∨ ¬p ⇔ **T**	**Or Tautology**
p ∧ ¬p ⇔ **F**	**And Contradiction**
(p→q) ⇔ (¬p ∨ q)	**Implication Equivalence**
p↔q⇔(p→q) ∧ (q→p)	**Biconditional Equivalence**

The Proof Process



Prove: (p∧¬q) ∨ q ⇔ p∨q

(p∧¬q) ∨ q	Left-Hand Statement
⇔ q ∨ (p∧¬q)	Commutative
⇔ (q∨p) ∧ (q ∨¬q)	Distributive
⇔ (q∨p) ∧ T	Or Tautology
⇔ q∨p	Identity
⇔ p∨q	Commutative

Begin with exactly the left-hand side statement
End with exactly what is on the right
Justify EVERY step with a logical equivalence

Predicate Calculus: Quantifiers

Universe of Discourse, U: The domain of a variable in a propositional function.

Universal Quantification of P(x) is the proposition:"P(x) is true for all values of x in U."

Existential Quantification of P(x) is the proposition: "There exists an element, x, in U such that P(x) is true."

Universal Quantification of P(x)

$\forall xP(x)$
"for all x P(x)"
"for every x P(x)"
Defined as:
$P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$ for all x_i in U

Example:
Let P(x) denote $x^2 \geq x$
If U is x such that $0 < x < 1$ then $\forall xP(x)$ is false.
If U is x such that $1 < x$ then $\forall xP(x)$ is true.

Existential Quantification of P(x)

$\exists xP(x)$
"there is an x such that P(x)"
"there is at least one x such that P(x)"
"there exists at least one x such that P(x)"
Defined as:
$P(x_0) \vee P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$ for all x_i in U
Example:
Let P(x) denote $x^2 \geq x$
If U is x such that $0 < x \leq 1$ then $\exists xP(x)$ is true.
If U is x such that $x < 1$ then $\exists xP(x)$ is true.

Quantifiers

$\forall xP(x)$
- True when P(x) is true for every x.
- False if there is an x for which P(x) is false.

$\exists xP(x)$
- True if there exists an x for which P(x) is true.
- False if P(x) is false for every x.

Negation (it is not the case)

$\neg\exists xP(x)$ equivalent to $\forall x\neg P(x)$
- True when P(x) is false for every x
- False if there is an x for which P(x) is true.

$\neg \forall xP(x)$ is equivalent to $\exists x\neg P(x)$
- True if there exists an x for which P(x) is false.
- False if P(x) is true for every x.

Examples 2a

Let T(a,b) denote the propositional function "a trusts b." Let U be the set of all people in the world.

Everybody trusts Bob.
$\forall xT(x,Bob)$
Could also say: $\forall x \in U\ T(x,Bob)$
$\in$ denotes membership
Bob trusts somebody.
$\exists xT(Bob,x)$

Examples 2b

Alice trusts herself.
T(Alice, Alice)

Alice trusts nobody.
$\forall x\ \neg T(Alice,x)$

Carol trusts everyone trusted by David.
$\forall x(T(David,x) \rightarrow T(Carol,x))$

Everyone trusts somebody.
$\forall x\ \exists y\ T(x,y)$

Quantification of Two Variables
(read left to right)

∀x∀yP(x,y) or ∀y∀xP(x,y)
•True when P(x,y) is true for every pair x,y.
•False if there is a pair x,y for which P(x,y) is false.

∃x∃yP(x,y) or ∃y∃xP(x,y)
True if there is a pair x,y for which P(x,y) is true.
False if P(x,y) is false for every pair x,y.

Quantification of Two Variables

∀x∃yP(x,y)
•True when for every x there is a y for which P(x,y) is true.
(in this case y can depend on x)
•False if there is an x such that P(x,y) is false for every y.

∃y∀xP(x,y)
•True if there is a y for which P(x,y) is true for every x.
(i.e., true for a particular y regardless (or independent) of x)
•False if for every y there is an x for which P(x,y) is false.

Note that order matters here
In particular, if ∃y∀xP(x,y) is true, then ∀x∃yP(x,y) is true.
However, if ∀x∃yP(x,y) is true, it is not necessary that ∃y∀xP(x,y) is true.

Examples 3a

Let L(x,y) be the statement "x loves y" where U for both x and y is the set of all people in the world.

Everybody loves Jerry.
∀xL(x,Jerry)

Everybody loves somebody.
∀x ∃yL(x,y)

There is somebody whom everybody loves.
∃y∀xL(x,y)

Examples 3b1

There is somebody whom Lydia does not love.
∃x¬L(Lydia,x)

Nobody loves everybody. (For each person there is at least one person they do not love.)
∀x∃y¬L(x,y)

There is somebody (one or more) whom nobody loves
∃y ∀x ¬L(x,y)